Unification of Kinded Infinite Trees

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Abstract

We study the problem of unifying infinite trees with variables subject to constraints on the trees they may be substituted for. The constraints are given by mappings from variables to trees, called kindings. In particular, we show that if a set of equations over a given kinding is unifiable, it has a most general unifier using only a subset of the variables in the set of equations and in the kinding. We present an efficient algorithm to extract the most general unifier of a regular (having a finite number of subtrees) and unifiable set of equations over a given kinding. The results extend those of Huet on unification of regular trees, and those of Ohori on unification of kinded finite trees.

1 The Problem

We follow Courcelle on infinite trees [2]. Assume F to be a ranked alphabet, and V a set of variables, such that $F \cap V = \emptyset$. Variables in V are considered symbols of rank 0. A *tree over a ranked alphabet* $F \cup V$ is a partial mapping, $u : N_+^* \to F \cup V$, from the set N_+^* of strings over positive integers into symbols in $F \cup V$, satisfying a tree domain condition [2].

Let $M^\infty(F \cup V)$ denote the set of all trees over $F \cup V$, also denoted by $M^\infty(F, V)$. To emphasize the cardinality of V, take $V_n = \{v_1, v_2, \ldots, v_n\}$. Alternative sets of variables are X and W. Write $\operatorname{root} u$ for $u(\varepsilon)$, the label of the root of tree u, where ε is the empty string. We consistently use letters $v, v', \ldots$ for variable trees (that is, trees such that $\operatorname{root} v \in V$), letters $t, t', \ldots$ for non-variable trees (that is, when $\operatorname{root} t \in F$), and letters $u, u', \ldots$ for either. Strings of trees $u_1, \ldots, u_n$ $(n \geq 0)$ are denoted by $\vec{u}$.

If $u_1, \ldots, u_n$ are n $(n \geq 0)$ trees, and the rank of the symbol $f \in F$ is n, we write $f(u_1, \ldots, u_n)$ for the tree u such that:

$$\begin{aligned} &u(\varepsilon) = f, \\ &u(i\alpha) = u_i(\alpha) \quad \text{if } 1 \leq i \leq n \text{ and } \alpha \in N_+^*, \\ &u(\alpha) \text{ is undefined otherwise.} \end{aligned}$$

A (first order) *substitution* is a total map $\tau : V \to M^\infty(G)$, for G a ranked alphabet. Given a substitution τ and a tree u, we obtain the tree τu as the result of the simultaneous substitution of tree τv for each occurrence of each variable $v \in V$ in u.

Now assume L to be a totally ordered ranked alphabet, and build the ranked alphabet R of *record symbols* as the set of strings of pairwise distinct labels in L. Define the rank of a record symbol $l_1 \ldots l_n$ $(n \geq 1)$ as $\sum_{i=1}^{n} \operatorname{rank} l_i$, and the rank of the empty string ε as 0. Henceforth, assume $R \subseteq F$.

A *record* is a partial map of finite domain $r : L \to (M^\infty(F,V))^*$ from the set of labels into strings of infinite trees. The record associated with a (non-variable) tree t in $M^\infty(F,V)$, notation $[t]$, is a map:

$$\begin{array}{ll} \{l_i \mapsto u_{1+\sum_{j=1}^{i-1} \operatorname{rank} l_j} \cdots u_{\sum_{j=1}^{i} \operatorname{rank} l_j} \mid 1 \leq i \leq n\} & \text{if } t = l_1 \cdots l_n(u_1, \ldots, u_m), \\ \emptyset & \text{if } \operatorname{root} t \in F - R. \end{array}$$

We now adapt the notions in Ohori [5] to infinite trees. *Kindings* are total maps $K : W \to M^\infty(G,W)$ such that $\operatorname{root} Kv \in R$ for all $v \in W$. Let t and u be two trees in $M^\infty(G,W)$. Say *t subsumes u under K*, notation $t \succ_K u$, if:

$$[t] \subseteq \begin{cases} [Ku] & \text{if } \operatorname{root} u \in W, \\ [u] & \text{otherwise.} \end{cases}$$

Let $\tau : V \to M^\infty(G,W)$ be a substitution. A *kinded substitution* is a pair (K, τ). Say (K,τ) *respects* $K' : V \to M^\infty(F,V)$ if $\tau K' v \succ_K \tau v$ for all $v \in V$.

Let E be a set of equations in $M^\infty(F,V) \times M^\infty(F,V)$; equations in E are written $u \doteq u'$. A *kinded set of equations* is a pair (K', E). Say (K,τ) is a *unifier* of (K', E) if it respects K', and $\tau u = \tau u'$ for all $u \doteq u' \in E$. Denote by $\operatorname{Unif}(K,E)$ the set of all unifiers of (K,E), and say (K,E) is *unifiable* if $\operatorname{Unif}(K,E) \neq \emptyset$.

Now let $K' : X \to M^\infty(F,X)$ be a kinding, and $\tau' : V \to M^\infty(F,X)$ a substitution. Say (K,τ) is *more general than* (K',τ') if there is a substitution $\tau'' : W \to M^\infty(F,X)$ such that $\tau' = \tau''\tau$, and (K',τ'') respects K.

It is easy to see that most general unifiers are unique up to a renaming of variables. Suppose that the two kinded substitutions (K,τ) and (K',τ') above are most general unifiers. There exists a $\theta : W \to M^\infty(F,X)$ such that $\tau' = \theta\tau$ and (K',θ) respects K. Similarly, there exists a $\theta' : X \to M^\infty(F,W)$ such that $\tau = \theta'\tau'$ and (K,θ') respects K'. It is not difficult to see that $\theta'\theta$ is the identity in W, and that X and W are in bijection by θ (see Courcelle [2] for details). As for the kindings, since (K,θ') respects K', we have $[\theta' K' v'] \subseteq [K\theta' v']$ for all $v' \in X$. Taking $v = \theta' v'$ (and thus $v' = \theta v$), we successively have: $[\theta' K' \theta v] \subseteq [Kv]$, $\theta[\theta' K' \theta v] \subseteq \theta[Kv]$, and $[K'\theta v] \subseteq [\theta K v]$. But, since (K',θ) respects K, we have $[\theta K v] \subseteq [K'\theta v]$ for all $v \in W$. Hence K' and K are in bijection by θ.

Regular trees are trees having finitely many different subtrees. A kinded set of equations (K,E) is regular if the trees in the equations in E, as well as those in the codomain of K, are regular. We are now a in position to state our main result.

Theorem 1 (Main Result) *Let E be a set of equations in $M^\infty(F,V_n) \times M^\infty(F,V_n)$, and K a kinding in $V_n \to M^\infty(F,V_n)$.*

i. If (K,E) is unifiable, it has a most general unifier (K',τ) with $K' : W_m \to M^\infty(F,W_m)$, $\tau : V_n \to M^\infty(F,W_m)$, and $m \leq n$.

ii. *If* (K, E) *is unifiable and regular, its most general unifier is regular. It can be effectively computed.*

We split the proof of the theorem into two parts each of which will be dealt in a separate section. The first concerns the *unifiability* of kinded sets of equations and enables to prove the first part of the theorem; the second concerns the *unification algorithm* and its correctness, and enables to prove the second part of the theorem.

2 Unifiability

For the rest of this section, let K be a kinding $V_n \to M^\infty(F, V_n)$, and E a set of equations in $M^\infty(F, V_n) \times M^\infty(F, V_n)$. Let $\mathcal{T}$ be the set of all subtrees in the equations in E and in the codomain of K.

We adapt Huet's notions of simplifiable and coherent relations [4] to the kinded setting. An equivalence relation $\sim$ on $\mathcal{T}$ is *simplifiable over a kinding* K if:

i. $f(\vec{u}) \sim f(\vec{u}') \;\Rightarrow\; \vec{u} \sim \vec{u}$,

ii. $v \sim v' \;\Rightarrow\; [Kv]l \sim [Kv']l$, for all $l \in \mathrm{dom}[Kv] \cap \mathrm{dom}[Kv']$,

iii. $v \sim t \;\Rightarrow\; [Kv]l \sim [t]l$, for all $l \in \mathrm{dom}[Kv] \cap \mathrm{dom}[t]$,

where $\vec{u} \sim \vec{u}'$ means $u_1 \sim u'_1, \ldots, u_n \sim u'_n$ if $\vec{u} = u_1, \ldots, u_n$ and $\vec{u}' = u'_1, \ldots, u'_n$, for some $n \geq 1$. Denote by $\sim_{KE}$ the least simplifiable relation on $\mathcal{T}$ over K containing E.

Proposition 2 $\mathrm{Unif}(K, E) = \mathrm{Unif}(K, \sim_{KE})$.

Proof. That $\mathrm{Unif}(K, E) \subseteq \mathrm{Unif}(K, \sim_{KE})$ follows from the definition of $\sim_{KE}$. The converse follows by a simple induction of the construction of $\sim_{KE}$. □

An equivalence relation $\sim$ on $\mathcal{T}$ is *coherent over a kinding* K if we do not have:

i. $f(\vec{u}) \sim g(\vec{u}')$ with $f \neq g$,

ii. $v \sim t$ with $\mathrm{dom}[Kv] \not\subseteq \mathrm{dom}[t]$.

To build the substitution $\sigma : V_n \to M^\infty(F, W)$ in the most general unifier of (K, E) we follow Courcelle [2]. Assume that $\sim_{KE}$ is coherent over K. Let X be the set of variables v in V_n such that:

i. $v \sim_{KE} t$, for some non-variable tree t,

ii. i. does not hold, $v = v_i$, and $v_i \sim_{KE} v_j$, for some $j > i$.

Let S the system of equations $\langle v \doteq u \mid v \in X \rangle$ where u is:

i. t if v satisfies i. above,

ii. v_j if v satisfies ii. above, where j is the largest index such that $v \sim_{KE} v_j$.

Let W be the set $V_n - X$; it is not difficult to prove that S is a generalized Greibach system over $F \cup W$ with set of unknowns X (see Courcelle [2] for details). Without loss of generality assume that $X = \{v_1, \ldots, v_m\}$ and that $W = \{v_{m+1}, \ldots, v_n\}$. Let $\langle u_1, \ldots, u_m \rangle$ denote the unique solution of S in $M^\infty(F, W)$, and let σ be the substitution such that $\sigma v_i = u_i$ if $v_i \in X$, and $\sigma v = v$ if $v \in W$.

To build the kinding K_p in the most general unifier of (K, E) we need an auxiliary notion. A *closure* of $K : V \to M^\infty(F, V)$ over an equivalence relation $\sim$ on $\mathcal{T}$ is a kinding $K^\sim : V \to M^\infty(F, V)$ such that:

$$K^\sim v = l_1 \cdots l_n(\vec{u}_1, \ldots, \vec{u}_n)$$

where $\{l_1, \ldots, l_n\} = \bigcup_{v \sim v'} \mathrm{dom}[Kv']$, $l_1 < \cdots < l_n$, and, for each $1 \leq i \leq n$, $\vec{u}_i = [Kv']l_i$, for some v' such that $v \sim v'$ and $l_i \in \mathrm{dom}[Kv']$. Notice that if $v \sim v'$ then $K^\sim v = K^\sim v'$, and also $\mathrm{dom}[Kv] \subseteq \mathrm{dom}[K^\sim v']$. The kinding $K_p : W \to M^\infty(G, W)$ is the restriction of the kinding $\sigma K^{\sim_{KE}}$ to the domain W.

We are now ready to present the main result concerning unifiability.

Lemma 3 (Unifiability)

i. (K, E) *is unifiable if and only if* $\sim_{KE}$ *is coherent.*

ii. If (K, E) *is unifiable, then* (K_p, σ) *is a most general unifier of* (K, E).

Proof. Write $\sim$ for $\sim_{KE}$. By proposition 2, we can prove the results with (K, E) replaced by $(K, \sim)$.

i. $(\Rightarrow)$ If $\sim$ is not coherent, then $\mathrm{Unif}(K, \sim) = \emptyset$.

$(\Leftarrow)$ Assume $\sim$ coherent. We show that (K_p, σ) unifies (K, E). Courcelle [2] proves that σ unifies E; we need to show that (K_p, σ) respects K, that is, $\sigma K v \succ_{K_p} \sigma v$ for all $v \in V$. We distinguish two cases. If there is a tree t such that $v \sim t$, make $r = [Kv]$ and $r' = [t]$. Since $\sim$ is simplifiable, $rl \sim r'l$, hence $\sigma r l = \sigma r' l$, for all $l \in \mathrm{dom}\, r \cap \mathrm{dom}\, r'$; since $\sim$ is coherent, $\mathrm{dom}\, r \subseteq \mathrm{dom}\, r'$, hence $\mathrm{dom}\, \sigma r \subseteq \mathrm{dom}\, \sigma r'$. Therefore, $[\sigma K v] \subseteq [\sigma t] = [\sigma v]$, and thus $\sigma K v \succ_{K_p} \sigma v$. Otherwise, from the construction of σ we know that $\sigma v = v_j$ where j is the largest index such that $v \sim v_j$. For each $l \in \mathrm{dom}[K^\sim v_j]$, we know from the construction of $K^\sim$ that there is a v' such that $v_j \sim v'$ and $[K^\sim v_j]l = [Kv']l$. Also, since $\sim$ is simplifiable, $[Kv]l \sim [Kv']l$, and thus $[\sigma K v]l = [\sigma K v']l = [\sigma K^\sim v_j]l = [K_p \sigma v]l$. Therefore, and since $\mathrm{dom}[Kv] \subseteq \mathrm{dom}[K^\sim v_j]$, we have $[\sigma K v] \subseteq [K_p \sigma v]$, and thus $\sigma K v \succ_{K_p} \sigma v$.

ii. Suppose that (K', τ) also unifies $(K, \sim)$; we show that (K_p, σ) is more general than (K', τ). Courcelle proves that there is a substitution τ' such that $\tau = \tau' \sigma$; we have to show that (K', τ') respects K_p, that is, $\tau' K_p v \succ_{K'} \tau' v$ for all $v \in W$. Since (K', τ) unifies $(K, \sim)$, we have $\tau K v \succ_{K'} \tau v$ for all $v \in V$. Hence $\tau K^\sim v \succ_{K'} \tau v$ for all $v \in W$. But $\tau K^\sim v = \tau' \sigma K^\sim v = \tau' K_p v$ and $\tau v = \tau' \sigma v = \tau' v$. Therefore $\tau' K_p v \succ_{K'} \tau' v$ for all $v \in W$. $\square$

3 A Unification Algorithm

In view of Lemma 3 we have to produce an algorithm $\mathcal{U}$ that returns $\sim_{KE}$ if $\sim_{KE}$ is coherent, and that signals failure otherwise. The notion of a canonical representative of a tree plays a fundamental role in the algorithm (cf. Huet [4]). A *representative of* *a tree* u under an equivalence relation $\sim$, notation $\mathrm{rep}_\sim u$, is a non-variable tree t if there is a t such that $t \sim u$, or a variable tree v such that $v \sim u$ otherwise.

Denote by $A^\sharp$ the reflexive, symmetric, and transitive closure of a binary relation A. Notation $E \cdot u \doteq u'$ means $E \cup \{u \doteq u'\}$ provided that $u \doteq u' \notin E$.

The algorithm $\mathcal{U}$ below is an adaptation to the kinded setting of Huet's algorithm [4], by using transformation rules in the style of Gallier and Snyder [3]. Equations are considered unordered, so that, strictly speaking, rule iv. comprises two rules.

Input: A set of equations E and a kinding K.

Initialization: Set E_0 equal to E, and $\sim_0$ equal the empty set.

Loop: Apply the transformation rules described below, until no rule applies.

i. If $u \sim u'$, then

$$E \cdot u \doteq u' \, , \, \sim \quad \longrightarrow \quad E \, , \, \sim .$$

ii. If i. does not hold, $f(\vec{u}) = \mathrm{rep}_\sim u$, and $f(\vec{u}') = \mathrm{rep}_\sim u'$, then

$$E \cdot u \doteq u' \, , \, \sim \quad \longrightarrow \quad E \cup \{\vec{u} \doteq \vec{u}'\} \, , \, (\sim \cup \{u \doteq u'\})^\sharp .$$

iii. If i. does not hold, $r = [K^\sim \mathrm{rep}_\sim v]$, and $r' = [K^\sim \mathrm{rep}_\sim v']$, then

$$E \cdot v \doteq v' \, , \, \sim \quad \longrightarrow \quad E \cup \{rl \doteq r'l \mid l \in \mathrm{dom}\, r \cap \mathrm{dom}\, r'\} \, , \, (\sim \cup \{v \doteq v'\})^\sharp .$$

iv. If i. does not hold, $r = [K^\sim \mathrm{rep}_\sim v], r' = [\mathrm{rep}_\sim u]$, and $\mathrm{dom}\, r \subseteq \mathrm{dom}\, r'$, then

$$E \cdot v \doteq u \, , \, \sim \quad \longrightarrow \quad E \cup \{rl \doteq r'l \mid l \in \mathrm{dom}\, r\} \, , \, (\sim \cup \{v \doteq u\})^\sharp .$$

Output: SUCCESS($\sim_n$) if $E_n = \emptyset$; FAIL otherwise.

When compared with the algorithm for kinded finite trees [5], the present algorithm saves two rules: one by treating record-trees no different from ordinary trees, the other by assigning the empty record to trees whose root is in $F - R$.

A few words on the complexity of the algorithm. Huet's original algorithm is known to be quasi-linear [4] on the number of subtrees in the equations, if $\sim$ is implemented with the union-find method [1] (with balanced trees and path compression). The closure of K can be maintained, initializing it to K, and updating it in rule iii., merging the components of the two records involved (ties broken arbitrarily), and assigning the new record to the representative of the variables involved. (No such operation is needed in rule iv. since, after the application of the rule, the representative of v is not a variable anymore.) If records are maintained in an explicit format, accessing their components, as well as computing the intersection and checking the inclusion of their domains is at most linear on the size of the domains involved. With this in mind, the algorithm runs in a time proportional to the product of the number of subtrees in the input and the time to perform the record operations.

If $\sim$ is implemented with the union-find method, then the union operation joins two equivalence classes, and the find operation finds the representative of a tree. In this case, care must be taken in rule iv. to join v to u and not vice-versa.

Lemma 4 (Termination) *If (K,E) is regular, $\mathcal{U}(K,E)$ terminates.*

Proof. If (K,E) is regular, $\sim_{KE}$ is finite. Clearly each $\sim$ is in $\sim_{KE}$. Rule i. reduces $|E|$, the number of equations in E; the remaining rules increase $|\sim|$, the size of $\sim$. We can easily prove that each rule strictly increases $r|\sim| - |E|$, where r is the largest rank of the symbols in F. $\square$

Lemma 5 (Invariant) *At each state n of the algorithm, we have:*

i. $\sim_n \subseteq (\bigcup_{i=0}^{n-1} E_i)^\sharp \subseteq (E_n \cup \sim_n)^\sharp$, *for* $n \geq 1$, *and*

ii. $(\bigcup_{i=0}^{n} E_i)^\sharp \subseteq \sim_{KE}$, *for* $n \geq 0$.

Proof. By induction on n.

i. When $n = 1$, suppose $E_0 = E' \cdot u \doteq u'$. Since $u \not\sim_0 u'$, we have $\sim_1 = \{u \doteq u'\}^\sharp \subseteq E_0^\sharp = (E_1 \cup \sim_1)^\sharp$. When $n > 1$, suppose $E_n = E' \cdot u \doteq u'$. By hypothesis $\sim_n \subseteq (\bigcup_{i=0}^{n-1} E_i)^\sharp \subseteq (E_n \cup \sim_n)^\sharp$. If $u \sim_n u'$, we have $\sim_{n+1} = \sim_n \subseteq (\bigcup_{i=0}^{n-1} E_i)^\sharp \subseteq (\bigcup_{i=0}^{n} E_i)^\sharp = (\bigcup_{i=0}^{n-1} E_i \cup E_n)^\sharp \subseteq ((E_n \cup \sim_n) \cup (E_{n+1} \cup \{u \doteq u'\}))^\sharp = (E_{n+1} \cup \sim_{n+1})^\sharp$. Otherwise $\sim_{n+1} = (\sim_n \cup \{u \doteq u'\})^\sharp \subseteq (\bigcup_{i=1}^{n-1} E_i \cup E_n)^\sharp = (\bigcup_{i=0}^{n} E_i)^\sharp = (\bigcup_{i=0}^{n-1} E_i \cup E_n)^\sharp \subseteq (E_n \cup \sim \cup E_n)^\sharp = (E' \cdot u \doteq u' \cup \sim_n)^\sharp \subseteq (E_{n+1} \cup \sim_{n+1})^\sharp$.

ii. When $n = 0$ we have $E_0^\sharp = E^\sharp \subseteq \sim_{KE}$. When $n > 0$ we have four cases depending on the rule applicable. For the first case, $E_{n+1} \subseteq E_n$. For the remaining cases, notice that, since $\sim_{KE}$ is simplifiable, the equations generated are in $\sim_{KE}$. $\square$

Clearly, upon a successful termination we have $\sim_n = (\bigcup_{i=0}^{n} E_i)^\sharp$. At termination we have the following result.

Lemma 6 *If no rule applies at state n ($n \geq 0$), then $\sim_{KE} \subseteq (\bigcup_{i=0}^{n} E_i)^\sharp$.*

Proof. By induction on the structure of $\sim_{KE}$. Clearly $E = E_0 \subseteq (\bigcup_{i=0}^{n} E_i)^\sharp$. Write $u \sharp_n u'$ when $u \doteq u' \in (\bigcup_{i=0}^{n} E_i)^\sharp$. Now let $u_1 \sim_{KE} u_2$; without loss of generality, suppose that the equation $u_1 \doteq u_2$ is selected at state j ($0 \leq j < n$). We have three cases according to the definition of simplifiable relation. In all of them, if rule i. applies, we have $u_1 \sim_j u_2$; and by the invariant $u_1 \sharp_{j-1} u_2$.

Otherwise, if $u_1 = f(\vec{u}_1)$, $u_2 = f(\vec{u}_2)$, we have to show that $\vec{u}_1 \sharp_n \vec{u}_2$. Rule ii. applies; let $\mathrm{rep}_{\sim_j} f(\vec{u}_1) = f(\vec{u}'_1)$, and $\mathrm{rep}_{\sim_j} f(\vec{u}_2) = f(\vec{u}'_2)$. Since $f(\vec{u}_1) \sim_j f(\vec{u}'_1)$, by the invariant $f(\vec{u}_1) \sim_{KE} f(\vec{u}'_1)$; since $\sim_{KE}$ is simplifiable, $\vec{u}_1 \sim_{KE} \vec{u}'_1$; and by induction $\vec{u}_1 \sharp_n \vec{u}'_1$. Similarly, $\vec{u}_2 \sharp_n \vec{u}'_2$. But $\vec{u}'_1 \doteq \vec{u}'_2 \in E_{j+1}$. Therefore $\vec{u}_1 \sharp_n \vec{u}_2$.

Now suppose that $u_1 = v$, $u_2 = t$; we have to show that $[Kv]l \sharp_n [t]l$ for all l in $\mathrm{dom}[Kv] \cap \mathrm{dom}[t]$. If $\mathrm{rep}_{\sim_j} v$ is a non-variable tree, apply rule ii. as above. Otherwise rule iv. applies; let $\mathrm{rep}_{\sim_j} v = v'$ and $\mathrm{rep}_{\sim_j} t = t'$. We have $[K^{\sim_j} v']l \doteq [t']l \in E_{j+1}$ for all l in $\mathrm{dom}[K^{\sim_j} v'] \subseteq \mathrm{dom}[t']$. For each of these l we know from the construction of $K^{\sim_j}$ that there is a v'' such that $v' \sim_j v''$ and $[K^{\sim_j} v']l = [Kv'']l$. Since $v' \sim_j v''$, following the reasoning for rule ii. above, we have $[Kv']l \sharp_n [Kv'']l$. Similarly, $[Kv]l \sharp_n [Kv'']l$ for all l in $\mathrm{dom}[Kv] \cap \mathrm{dom}[Kv'']$, and $[t]l \sharp_n [t']l$ for all l in $\mathrm{dom}[t] = \mathrm{dom}[t']$. Hence $[Kv]l \sharp_n [t]l$ for all l in $\mathrm{dom}[Kv] \cap \mathrm{dom}[t]$.

The case where rule iii. applies is easier. $\square$

Clearly, at termination we have $\sim_{KE} = (\bigcup_{i=0}^{n} E_i)^\sharp$. We may now state and prove the main result concerning the unification algorithm.

Lemma 7 (Correctness of $\mathcal{U}$) *Let (K,E) be a regular kinded set of equations.*

i. $\mathcal{U}(K,E)$ *succeeds if and only if* $\sim_{KE}$ *is coherent.*

ii. If $\mathcal{U}(K,E)$ *succeeds, it produces* $\sim_{KE}$.

Proof. By lemma 4, $\mathcal{U}(K,E)$ terminates; let n ($n \geq 0$) be the last state.

i. ($\Rightarrow$) Suppose that $\sim_{KE}$ is not coherent. Then, there is a pair of trees u, u' in $\sim_{KE}$ such that *either* $u = f(\vec{u})$, $u' = g(\vec{u}')$, and $f \neq g$, *or* $\operatorname{root} u \in V$, $\operatorname{root} u' \in F$, and $\operatorname{dom}[Ku] \not\subseteq \operatorname{dom}[u']$. By lemma 6, $\sim_{KE} \subseteq (\bigcup_{i=0}^{n} E_i)^\sharp$; without loss of generality suppose that equation $u \doteq u' \in E_j$ for some j ($0 \leq j \leq n$). Since no rule applies to this equation, it must be in E_n, and hence $\mathcal{U}(K,E)$ does not succeed.

($\Leftarrow$) Suppose that $\mathcal{U}(K,E)$ fails. Then E_n is nonempty, and since no rule applies to E_n, this must contain an equation $u \doteq u'$ with $u \not\sim_n u'$ and *either* $\operatorname{rep}_{\sim_n} u = f(\vec{u})$, $\operatorname{rep}_{\sim_n} u' = g(\vec{u}')$, and $f \neq g$, *or* $\operatorname{rep}_{\sim_n} u = v$, $\operatorname{rep}_{\sim_n} u' = t$, and $\operatorname{dom}[K^{\sim_n} v] \not\subseteq \operatorname{dom}[t]$. In either case $E_n^\sharp$ is not coherent, and by lemma 5.ii. $\sim_{KE}$ is not coherent.

ii. If $\mathcal{U}(K,E)$ succeeds, $E_n = \emptyset$. By lemmas 5 and 6, $\sim_n = \sim_{KE}$. □

Proof of the Main Result. *i.* The first part follows directly from lemma 3.ii. The second part follows from the fact that $W \subseteq V_n$.

ii. For the first part, notice that if (K,E) is regular, then $\mathcal{T}$ is a set of regular trees, and S an extended regular system whose solution consists of regular trees, and thus both σ and K_p are regular. For the second part, notice that if (K,E) is unifiable, by lemma 3.i, $\sim_{KE}$ is coherent, and, since (K,E) is regular, by lemma 7, $\mathcal{U}(K,E)$ succeeds producing $\sim_{KE}$. □

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